

New method to integrate 2+1 wave equations with Dirac's delta functions as sources

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Abstract

Gravitational perturbations in a Kerr black hole background can not be decomposed into simple tensor harmonics in the time domain. Here, we make the mode decomposition only in the azimuthal direction and discuss the resulting (2+1)-dimensional Klein-Gordon differential equation for scalar perturbations with a two dimensional Dirac's δ -function as a source representing a point particle orbiting a much larger black hole. To make this equation amenable for numerical integrations we explicitly remove analytically the singular behavior of the source and compute a global effective source for the corresponding waveform.

1 Introduction

One of the main source targets of LISA is the gravitational waves generated by the inspiral of compact objects into massive black holes. For these Extreme Mass Ratio Inspirals (EMRI) we use the black hole perturbation approach to compute waveforms, where the compact object is approximated by a point particle orbiting a massive Kerr black hole. In order to obtain the precise theoretical gravitational waveform, we need to solve the self-force problem and problems in the second order perturbations [1].

In this paper, we focus on one aspect of the self-force problem, specifically to derive the retarded field of a point source. As a first step, we consider the Klein-Gordon equation in the Schwarzschild spacetime, but do not decompose it into spherical harmonics, in order to model perturbations in the more generic Kerr background. Recently, Barack and Gollbourn [2] have discussed this equation in (2+1)-dimensions as derived by the mode decomposition in the azimuthal direction. Another treatment is proposed here to deal with this problem globally. We also note that there is a method to use a narrow Gaussian rather than a Dirac δ -function [3]. It is, however, difficult to ascertain the error introduced by smearing the particle and if this is accurate enough for self force computations.

2 Formulation

When we calculate the (2 + 1)-dimensional (D) equation derived from the 4D Klein-Gordon equation by the azimuthal mode decomposition, the resulting equation is not exactly same as the (2 + 1) D wave equation. By transforming the scalar field, we can obtain an equation which includes the (2 + 1) D d'Alambertian and a remainder. Then, we remove the 2D δ -function in the source term.

In order to do so, we consider the Schwarzschild metric in the isotropic coordinates,

$$ds^2 = -(2\rho - M)^2/(2\rho + M)^2 dt^2 + (1 + M/(2\rho))^4 (d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)) . \quad (1)$$

This radial coordinate is related to that of the Schwarzschild, r , as $\rho = (r - M + \sqrt{r^2 - 2Mr})/2$. In the above coordinates, the Klein-Gordon equation with a point source becomes

$$\left[-\frac{(2\rho + M)^2}{(2\rho - M)^2} \partial_t^2 + \frac{16\rho^4}{(2\rho + M)^4} \partial_\rho^2 + \frac{128\rho^5}{(2\rho - M)(2\rho + M)^5} \partial_\rho + \frac{16\rho^2}{(2\rho + M)^4} \left(\partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \right] \times \psi(t, \rho, \theta, \phi) = -q \int_{-\infty}^{\infty} d\tau \frac{64\rho^4 \delta(t - t_z(\tau)) \delta(\rho - \rho_z(\tau)) \delta(\theta - \theta_z(\tau)) \delta(\phi - \phi_z(\tau))}{(2\rho - M)(2\rho + M)^5 \sin \theta} . \quad (2)$$

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Here, we use the azimuthal mode decomposition, $\psi(t, r, \theta, \phi) = \sum_{m=-\infty}^{\infty} \psi_m(t, \rho, \theta) e^{im\phi}$, and transform the field ψ_m as $\psi_m = 2\sqrt{\rho/[(2\rho + M)(2\rho - M)\sin\theta]} \chi_m$. Then, χ_m satisfies the following equation.

$$\left[-\frac{1}{16} \frac{(2\rho + M)^6}{(2\rho - M)^2 \rho^4} \partial_t^2 + \partial_\rho^2 + \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \left(\partial_\theta^2 - \frac{1}{\sin^2 \theta} \left(m^2 - \frac{1}{4} \frac{(4\rho^2 + M^2)^2 - 16\rho M \cos^2 \theta}{(2\rho + M)^2 (2\rho - M)^2} \right) \right) \right] \times \chi_m(t, \rho, \theta) = -q \int_{-\infty}^{\infty} d\tau \frac{2\delta(t - t_z(\tau))\delta(\rho - \rho_z(\tau))\delta(\theta - \theta_z(\tau))}{\sqrt{(2\rho - M)(2\rho + M)\rho \sin \theta}} e^{-im\phi_z(\tau)}. \quad (3)$$

Next, we use a new time coordinate which is defined by $T = \int^t dt 4(2\rho_z(t) - M)\rho_z(t)^2/(2\rho_z(t) + M)^3$, where ρ_z is obtained by solving a geodesic equation. Using this, we derive an equation which can be divided into the $(2+1)D$ d'Alembertian of the flat case $\square^{(2+1)}$ and a remainder.

$$\begin{aligned} \mathcal{L}_m \chi_m(T, \rho, \theta) &= \left(\square^{(2+1)} + \mathcal{L}_m^{rem} \right) \chi_m(T, \rho, \theta) = S_m(T, \rho, \theta); \\ \square^{(2+1)} &= -\partial_T^2 + \partial_\rho^2 + (1/\rho)\partial_\rho + (1/\rho^2)\partial_\theta^2, \\ \mathcal{L}_m^{rem} &= \left(1 - \frac{(2\rho_z(T) - M)^2 \rho_z(T)^4 (2\rho + M)^6}{(2\rho_z(T) + M)^6 (2\rho - M)^2 \rho^4} \right) \partial_T^2 \\ &\quad - \frac{2(4\rho_z(T) - M)(2\rho_z(T) - M)(2\rho + M)^6 \rho_z(T)^3 M}{(2\rho_z(T) + M)^7 (2\rho - M)^2 \rho^4} \left(\frac{d\rho_z(T)}{dT} \right) \partial_T \\ &\quad - \frac{1}{\rho^2 \sin^2 \theta} \left(m^2 - \frac{1}{4} \frac{(4\rho^2 + M^2)^2 - 16\rho M \cos^2 \theta}{(2\rho + M)^2 (2\rho - M)^2} \right), \\ S_m(T, \rho, \theta) &= -q \int_{-\infty}^{\infty} d\tau \frac{2\delta(t(T) - t_z(\tau))\delta(\rho - \rho_z(\tau))\delta(\theta - \theta_z(\tau))}{\sqrt{(2\rho - M)(2\rho + M)\rho \sin \theta}} e^{-im\phi_z(\tau)}. \end{aligned} \quad (4)$$

To remove the δ -function in the source term, we set

$$\chi_m(T, r, \theta) = \chi_m^S(T, r, \theta) + \chi_m^{rem}(T, r, \theta), \quad (5)$$

where we define the new functions, χ_m^S and χ_m^{rem} as calculated from

$$\square^{(2+1)} \chi_m^S(T, r, \theta) = S_m(T, r, \theta); \quad \mathcal{L}_m \chi_m^{rem}(T, r, \theta) = -\mathcal{L}_m^{rem} \chi_m^S(T, r, \theta) = S_m^{(eff)}(T, \rho, \theta). \quad (6)$$

This decomposition of χ_m does not have any physical-meaning, i.e., χ_m^S is not identified as the singular part to be removed in the self-force calculation. Note that \mathcal{L}_m^{rem} includes a second-order derivative. But, since the factor of ∂_T^2 is zero at the particle location, the singular behavior of the effective source $S_m^{(eff)}$ weakens. The derivation of the singular field χ_m^S can be performed through the Green function,

$$G(T, \mathbf{x}; T', \mathbf{x}') = \theta((T - T') - |\mathbf{x} - \mathbf{x}'|) / (2\pi \sqrt{(T - T')^2 - |\mathbf{x} - \mathbf{x}'|^2}), \quad (7)$$

where $|\mathbf{x} - \mathbf{x}'| = (\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta'))^{1/2}$ and χ_m^S is calculated by the following integral

$$\chi_m^S(T, \rho, \theta) = \int dT' \rho' d\theta' G(T, \mathbf{x}; T', \mathbf{x}') S_m(T', \rho', \theta'). \quad (8)$$

3 Circular Orbit Case

We consider a particle in circular orbit, $z^\alpha(\tau) = \{u^t \tau, r_0, \pi/2, u^\phi \tau\}$, where $u^t = \sqrt{r_0/(r_0 - 3M)}$ and $u^\phi = \sqrt{M/[r_0^2(r_0 - 3M)]}$. Here, we focus on the $m \neq 0$ modes; the $m = 0$ mode can be dealt with by the same treatment.

3.1 Singular field

First, the relationship between the new time coordinate T and the Schwarzschild time t is obtained analytically as $T = 4(2\rho_0 - M)\rho_0^2/(2\rho_0 + M)^3 t$ where $\rho_0 = 1/2(r_0 - M + \sqrt{r_0^2 - 2Mr_0})$. Note that in

general, for non circular orbits, we need a numerical integration to derive this relationship. From Eq. (8), the singular field is derived as

$$\chi_m^S(t, r, \theta) = \frac{i}{4} \frac{2\sqrt{\rho_0}}{u^t \sqrt{(2\rho_0 + M)(2\rho_0 - M)}} H_0^{(1)} \left(\frac{(2\rho_0 + M)^3}{4(2\rho_0 - M)\rho_0^2} m\Omega |\mathbf{x} - \mathbf{x}_z| \right) e^{-im\Omega t}, \quad (9)$$

where $\Omega = u^\phi/u^t$ and the spatial difference is given by $|\mathbf{x} - \mathbf{x}_z| = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \sin\theta}$ and $H_0^{(1)}$ is the Hankel function. The local behavior of the above solution near the particle location is obtained as

$$\chi_m^S(t, r, \theta) \sim \chi_m^{SL}(t, r, \theta) = -\frac{1}{2\pi} \frac{2\sqrt{\rho_0}}{u^t \sqrt{(2\rho_0 + M)(2\rho_0 - M)}} \ln \left(\frac{(2\rho_0 + M)^3}{4(2\rho_0 - M)\rho_0^2} m\Omega |\mathbf{x} - \mathbf{x}_z| \right) e^{-im\Omega t}. \quad (10)$$

Therefore, $S_m^{(eff)}(t, \rho, \theta)$ shown by the dashed green curve in Fig. 1, is singular at the particle location. In order to perform the numerical integration with higher accuracy, it is convenient to regularize the source term to be at least C^0 at the particle location.

3.2 Local behavior

When we write the singular field as $\chi_m^S = \chi_m^{SL} + \hat{\chi}_m^S$, $\hat{\chi}_m^S$ is finite at the particle location. Then, the effective source in Eq. (6) becomes

$$\begin{aligned} S_m^{(eff)}(t, \rho, \theta) &= -\frac{1}{2\pi} \frac{2\sqrt{\rho_0}}{u^t \sqrt{(2\rho_0 + M)(2\rho_0 - M)}} \ln \left(\frac{(2\rho_0 + M)^3}{4(2\rho_0 - M)\rho_0^2} m\Omega |\mathbf{x} - \mathbf{x}_z| \right) e^{-im\Omega t} \\ &\times \left(-\frac{1}{\rho^2 \sin^2 \theta} \left(m^2 - \frac{1}{4} \frac{(4\rho^2 + M^2)^2 - 16\rho M \cos^2 \theta}{(2\rho + M)^2 (2\rho - M)^2} \right) \right. \\ &\quad \left. - (m\Omega)^2 \left(\frac{(2\rho_0 + M)^6}{(2\rho_0 - M)^2 \rho_0^4} - \frac{(2\rho + M)^6}{(2\rho - M)^2 \rho^4} \right) \right) - \mathcal{L}_m^{rem} \hat{\chi}_m^S(t, r, \theta), \end{aligned} \quad (11)$$

Note that the third line of the R.H.S. is at least C^0 at the location of the particle.

To remove the logarithmic divergence in the source, we introduce

$$\begin{aligned} \chi_m^{rem,S}(t, \rho, \theta) &= -\frac{1}{16} |\mathbf{x} - \mathbf{x}_z|^2 \ln \left(\frac{(2\rho_0 + M)^3}{4(2\rho_0 - M)\rho_0^2} m\Omega |\mathbf{x} - \mathbf{x}_z| \right) \rho_0^{19/2} (2\rho - M)^3 e^{-im\Omega t} \\ &\times \frac{64m^2\rho^4 - 32m^2\rho^2M^2 + 4m^2M^4 + 16\cos^2\theta\rho^2M^2 - (4\rho^2 + M^2)^2}{\pi u^t (2\rho_0 + M)^{5/2} (2\rho_0 - M)^{11/2} \rho^{11} \sin^2 \theta}. \end{aligned} \quad (12)$$

Using this regularization function, we obtain a source $S_m^{reg,I}$ for the function $\chi_m^{rem} - \chi_m^{rem,S}$ as

$$S_m^{reg,I}(t, \rho, \theta) = S_m^{(eff)}(t, \rho, \theta) - \mathcal{L}_m \chi_m^{rem,S}(t, \rho, \theta). \quad (13)$$

The local behavior of $S_m^{reg,I}$, which is shown by the solid green curve in Fig. 1 (b), is an " $x \ln|x|$ " type, i.e., C^0 around the particle location.

3.3 Boundary behaviors

We now focus on the behaviors of the source term at the two boundaries, i.e at the horizon of the large hole and spatial infinity. The source for a final regularized function χ_m^{reg} must go like $O(\rho^{-2})$ for $\rho \rightarrow \infty$ in the case of the $m = 0$ mode and $O(\rho^{-3/2})$ for the $m \neq 0$ mode because of integrability conditions. More precisely, the source for the regularized function of ψ_m derived by numerical calculations has a factor $\sim 1/\rho^{1/2}$. For $\rho \rightarrow M/2$, the source should be zero, i.e., the behavior should be a power of $(\rho - M/2)$ greater than $1/2$. To regularize the source at the boundaries, we note that the source contribution from $\chi_m^{rem,S}$ is well behaved. This means that the ill behaviors of the source arise from χ_m^S . Therefore, it is convenient to use asymptotic behaviors of χ_m^S (and some correction factor) for regularization.

First, for the regularization near the horizon, we use the regularization function χ_m^h which is too long to be shown here. Then, the source for the function $\chi_m^{rem} - \chi_m^{rem,S} - \chi_m^h$ becomes

$$S_m^{reg,h}(t, \rho, \theta) = S_m^{reg,I}(t, \rho, \theta) - \mathcal{L}_m \chi_m^h(t, \rho, \theta), \quad (14)$$

This $S_m^{reg,h}$ is shown by the solid black curve in Fig. 1 and behaves as $O(\rho^{-1/2})$ for large ρ . To regularize it, we use the regularization function,

$$\begin{aligned} \chi_m^\infty(t, \rho, \theta) = & -\sqrt{2}i \frac{\rho_0^{3/2}}{(2\rho_0 + M)^2 u^t \sqrt{\pi} \sqrt{m\Omega\rho} \rho^7} e^{(-im\Omega t)} \left(\rho - \frac{M}{2}\right)^3 (\rho^2 + \rho_0^2 - 2\rho_0\rho \sin\theta)^2 \\ & \times \exp\left(\frac{1}{4}i \frac{(2\rho_0 + M)^3 m\Omega \sqrt{\rho^2 + \rho_0^2 - 2\rho_0\rho \sin\theta}}{(2\rho_0 - M)\rho_0^2} - \frac{\pi}{4}i\right). \end{aligned} \quad (15)$$

The final source for the regularized function $\chi_m^{reg} = \chi_m^{rem} - \chi_m^{rem,S} - \chi_m^h - \chi_m^\infty$ becomes

$$S_m^{reg,f}(t, \rho, \theta) = S_m^{reg,h}(t, \rho, \theta) - \mathcal{L}_m \chi_m^\infty(t, \rho, \theta), \quad (16)$$

which is used in the numerical calculation. This $S_m^{reg,f}$ is shown by the dashed black curve in Fig. 1, and behaves like $O(\rho^{-3/2}) \times$ (an oscillation factor with respect to ρ) for large ρ .

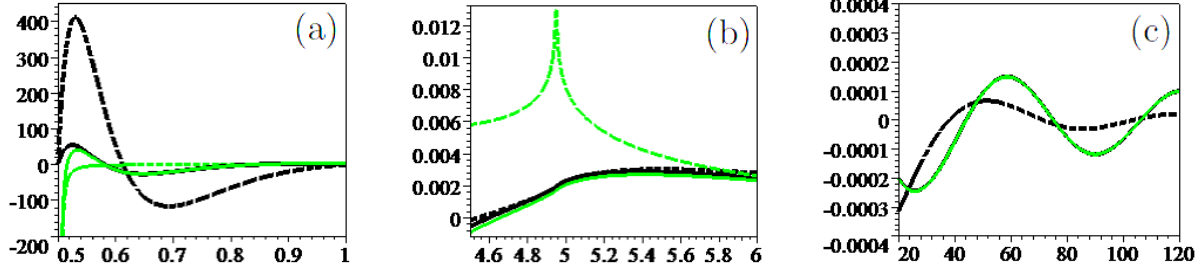


Figure 1: Plot for the $m = 1$ mode of S_m with respect to ρ . $S_1^{(eff)}$, $S_1^{reg,I}$, $S_1^{reg,h}$ and $S_1^{reg,f}$ are shown by the dashed green, solid green, solid black and dashed black curve, respectively.

4 Discussion

In this paper, we obtained the regularized effective source which is C^0 at the location of the particle, and $O(\rho - M/2)$ near the horizon. The behavior at infinity is $O(\rho^{-3/2}) \times$ (an oscillation factor with respect to ρ) which allows straightforward numerical integration.

When we consider the extension of this formulation to the Kerr background case, we can also extract a similar differential operator to that of Eq. (4). In the case of gravitational perturbations, we have ten field equations for the linear perturbation in the Lorenz gauge. (See [4].) The same treatment discussed in this paper is applicable to those equations.

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